TRINITY COLLEGE

YEAR 12
MATHEMATICS
SPECIALIST

Test 1, 2023
Section One: Calculator Free
Complex Numbers and Functions

STUDENT'S NAME: Solutions [LAWRENCE]

DATE: Thursday 16th March **TIME:** 40 minutes **MARKS:** 40
ASSESSMENT %: 10

INSTRUCTIONS:
Standard Items: Pens, pencils, drawing templates, eraser
Special Items:

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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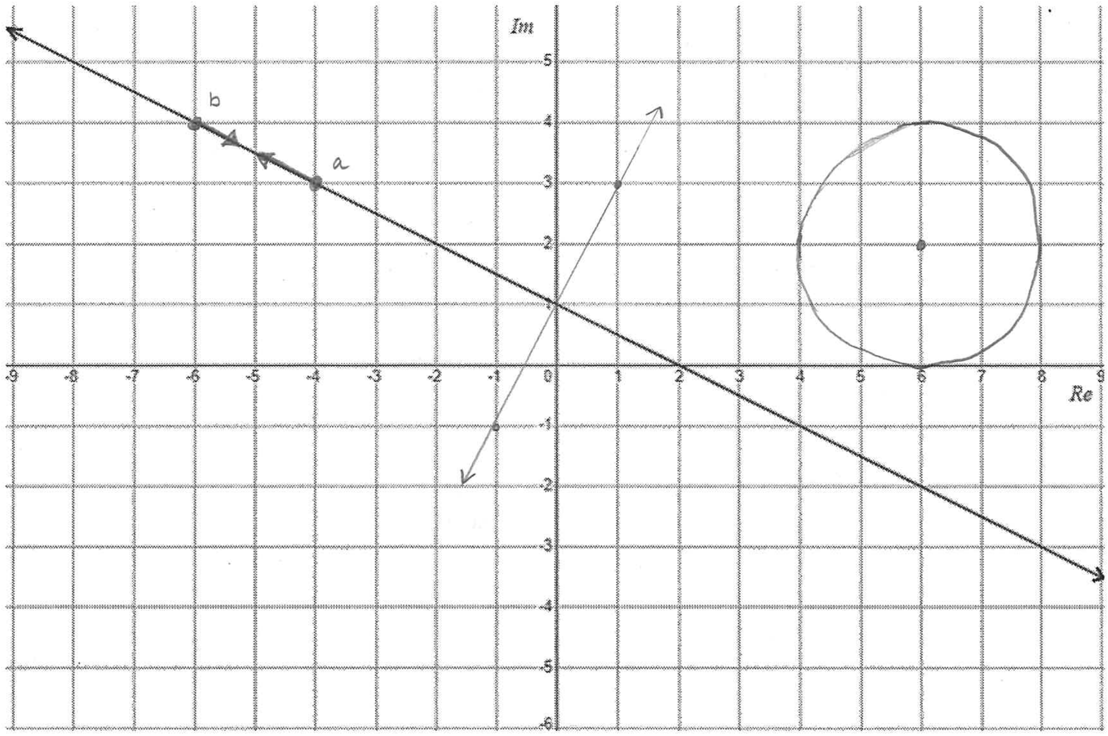
Question 1 **(10 marks)**

Consider the following complex locus for z.



(a) Represent the above locus using, but not necessarily limited to, the following.

i) Using the Absolute Value/Magnitude function. (2 marks)

$m = \frac{-1}{2}$

$\therefore$ m for ⊥ bisector = 2

$|z - (1+3i)| = |z - (-1-i)|$

(or any line // to the above)
(looking for a ⊥ bisector)

✓ correct 2 pts on a line
✓ ⊥ bisector

ii) Using $Re(z)$ and/or $Im(z)$. (1 mark)

$y = -\frac{1}{2}x + 1$

$Im(z) = \frac{-1}{2} Re(z) + 1$

✓ correct equation

iii) Using $Arg(z + a)$, $Arg(z + b)$ and ∪ (union), where $a, b \in \mathbb{C}$. (2 marks)

e.g. $Arg(z - (-4+3i)) \cup Arg(z - (-6+4i))$

✓ correct angles of lines & a point on the line used.

✓ whole line is represented.

(Many solutions)

Consider the following locus $|w-6-2i|=2$.

(b) Sketch the locus on the diagram provided at the beginning of this question. (2 marks)

$|w-(6+2i)|=2$

Circle centre $(6,2)$

radius $=2$

✓ centre

✓ radius, shape

(c) Determine the maximum and minimum values of $Arg(w)$.
Include a sketch in your working out to aid your response. (3 marks)

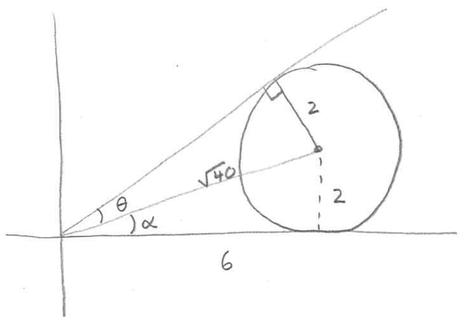
Min $Arg(w) = 0$

Max $Arg(w)$

✓ correct Min Arg

✓ correct θ & α

✓ correct Max Arg



$|w| = \sqrt{36+4}$
$= \sqrt{40}$
$= 2\sqrt{10}$

$\tan\alpha = \frac{1}{3}$

$\alpha = \tan^{-1}\left(\frac{1}{3}\right)$

$\sin\theta = \frac{2}{\sqrt{40}} = \frac{2}{2\sqrt{10}} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$

$\theta = \sin^{-1}\frac{\sqrt{10}}{10}$

$\therefore$ Max $Arg(w) = \tan^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{\sqrt{10}}{10}\right)$

Question 2 **(3 marks)**

Prove and state the conditions of $|z|$ and $Arg(z)$ for which $\bar{z} = z^{-1}$, $z \in \mathbb{C}$.

$z = x + yi$

$\bar{z} = x - yi$

$z^{-1} = \dfrac{1}{x+yi}$

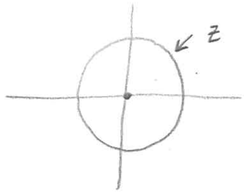
$x - yi = \dfrac{1}{x+yi}$

$(x-yi)(x+yi) = 1$

$x^2 + y^2 = 1$

$\therefore$ circle centre $(0,0)$

radius $= 1$



$Arg(z) \in \mathbb{R}$

$|z| = 1$

✓ represent $\bar{z}$ & z^{-1}

✓ simplifies to circle

✓ state conditions for $Arg(z)$ & $|z|$

Question 3 **(5 marks)**

Consider the function $f(z) = z^4 - 4z^3 + 9z^2 - 16z + 20$ where $z \in \mathbb{C}$.
Solve $f(z) = 0$ if $z - (2 + i)$ is a factor of $f(z)$.

$z - (2+i)$ is a factor

$\therefore$ $z - (2-i)$ is also a factor (conjugate)

$$(z-(2+i))(z-(2-i)) = z^2 - z(2+i) - z(2-i) + (2+i)(2-i)$$
$$= z^2 - 2z - \cancel{iz} - 2z + \cancel{iz} + 4 + 1$$
$$= z^2 - 4z + 5$$

$$\begin{array}{r|l} & z^2 + 4 \\ \hline z^2 - 4z + 5 & z^4 - 4z^3 + 9z^2 - 16z + 20 \\ - & z^4 - 4z^3 + 5z^2 \\ \hline & 4z^2 - 16z + 20 \\ & 4z^2 - 16z + 20 \\ \hline & 0 \end{array}$$

So, $z^4 - 4z^3 + 9z^2 - 16z + 20 = (z-(2+i))(z-(2-i))(z^2+4) = 0$

$z - (2+i) = 0$ $\quad$ $z - (2-i) = 0$ $\quad$ $z^2 + 4 = 0$

$z = 2 + i$ $\quad$ $z = 2 - i$ $\quad$ $z^2 = -4$

$z = \pm\sqrt{-4}$

$z = \pm 2i$

✓ states 2nd factor (conjugate)

✓ expands factor x factor

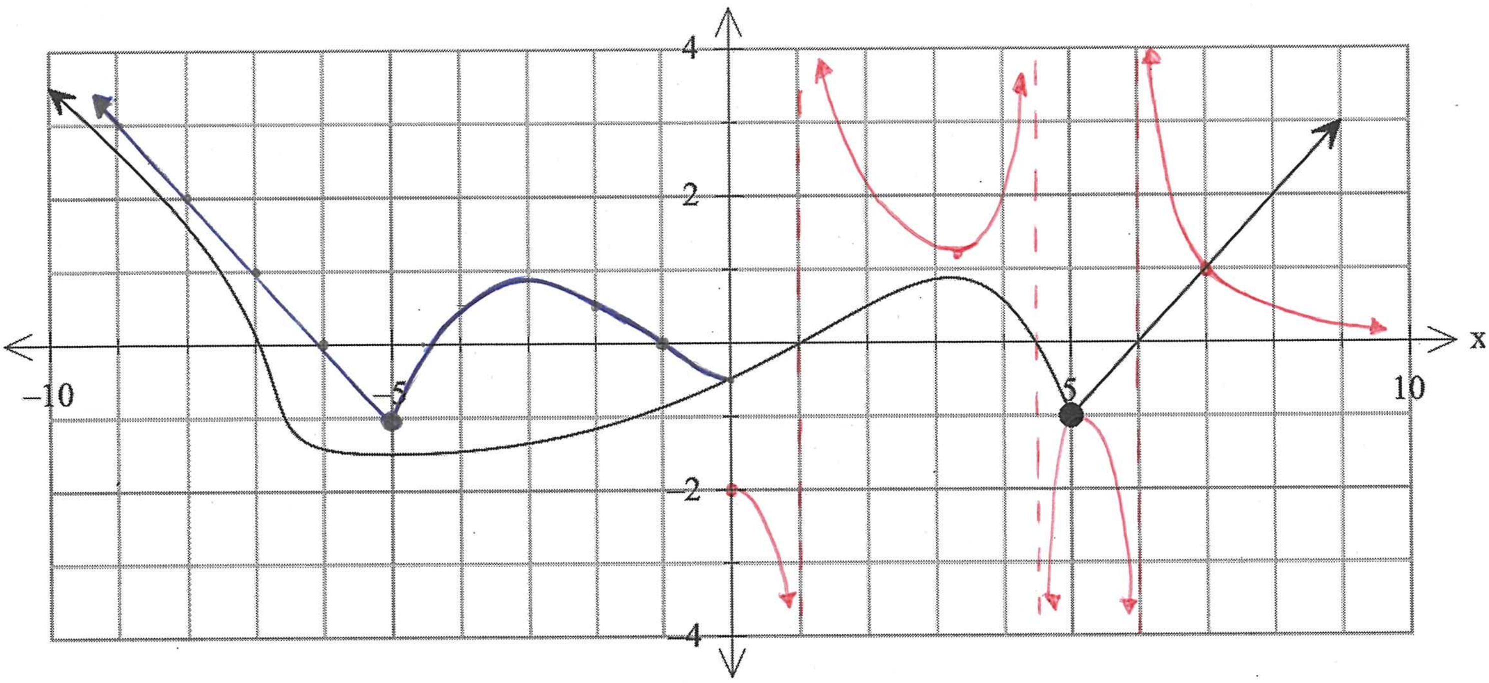
✓ divides to find 3rd factor

✓ states 1st 2 solutions

✓ states 2 solutions to 3rd factor

Question 4 **(9 marks)**

Consider the function $f(x)$ which has been sketched on the provided axis.



(a) On the above axis sketch $f(|x|)$ over the domain $x \leq 0$. In Blue (2 marks)

✓ reflected over $x = 0$

✓ correct shape

(b) On the above axis sketch $\frac{1}{f(x)}$ over the domain $x \geq 0$. In Red (3 marks)

✓ all vertical asymptotes

✓ shapes

✓ key points $(0,-2)$, $(7,1)$, $(5,-1)$,

(c) State when $f(x) = |f(x)|$. only when $f(x) \geq 0$ (1 mark)

$\therefore \quad x \leq -7$

$1 \leq x \leq 4.5$

$x \geq 6$

✓ all domains

(d) $g(x) = f(x)$ for the values $5 \leq x \leq 6$.
If $g(x) = -|ax + b| + c$, determine the values of a, b and c. (3 marks)

$g(x)$

$f(x)$ $m = 1$

$\therefore$ Cusp must be at $(6,0)$
and $g(x)$ & $f(x)$ must have same gradient

$\therefore \quad a = 1$ ✓ correct a

$b = -6$ ✓ b

$c = 0$ ✓ c

Question 5 **(3 marks)**

Calculate and state the nature of all asymptotes to the equation $f(x) = \frac{x^2+4x-7}{x-1}$.

Vertical asymptote @ $x = 1$

Check for Horizontal asymptote

$\lim_{x \to \infty} \frac{x^2}{x} (x) \Rightarrow \infty$

$\lim_{x \to -\infty} \frac{x^2}{x} (x) \Rightarrow -\infty$

$\therefore$ No Horizontal asymptote.

Improper Fraction $\therefore$ check for oblique asymptote

$$\begin{array}{r|l} & x + 5 \\ \hline x - 1 & x^2 + 4x - 7 \\ - & x^2 - x \\ \hline & 5x - 7 \\ - & 5x - 5 \\ \hline & -2 \end{array}$$

$f(x) = x + 5 - \frac{2}{x-1}$

$\therefore$ Oblique asymptote @ $y = x + 5$

✓ correct vertical

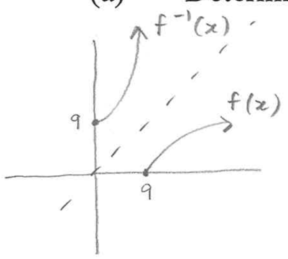
✓ divides polynomial

✓ states oblique asymptote

Question 6 **(10 marks)**

Consider the two equations: $f(x) = \sqrt{x-9}$ and $g(x) = -ax^2 + c \quad a, c \in \mathbb{R}, \ a \geq 0, \ c \geq 9.$

(a) Determine $f^{-1}(x)$ and state the domain and range. (3 marks)



$f^{-1}(x) = x^2 + 9$

$\{x : x \geq 0\}$

$\{y : y \geq 9\}$

✓ correct $f^{-1}(x)$

✓ states domain

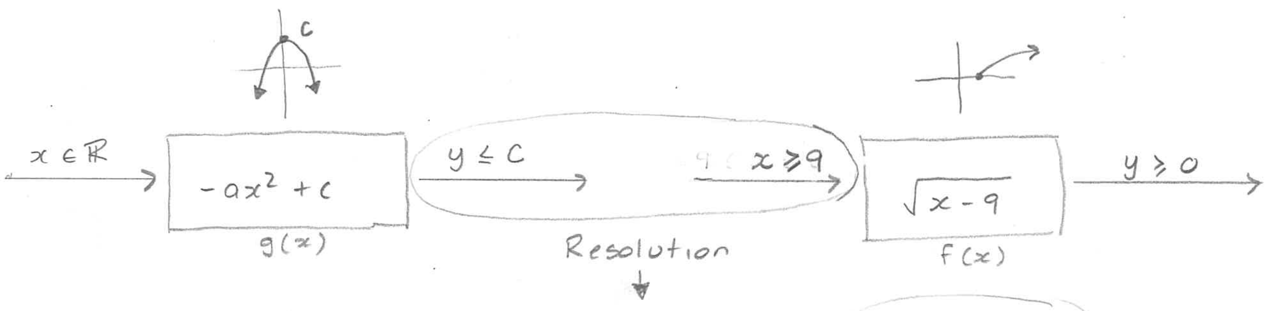
✓ states range

(b) Determine $f(g(x))$. (1 mark)

$f(g(x)) = \sqrt{-ax^2 + c - 9}$

✓

(c) Show and briefly explain how the generalised range of $f(g(x))$ is $0 \leq y \leq \sqrt{c-9}$. (2 marks)



$9 \leq x \leq c$

Min $\sqrt{9-9} = 0$

Max $\sqrt{c-9}$

$\therefore \quad 0 \leq y \leq \sqrt{c-9}$

✓ shows justification for Min 0

✓ shows justification for Max $\sqrt{c-9}$

(d) Show and briefly explain how the generalised domain of $f(g(x))$ is (2 marks)

$$\frac{-\sqrt{4a(c-9)}}{2a} \le x \le \frac{\sqrt{4a(c-9)}}{2a}$$

$f(g(x)) = \sqrt{-ax^2 + c - 9}$

✓ states inequality

Input must be ≥ 0

✓ shows use of quadratic formula

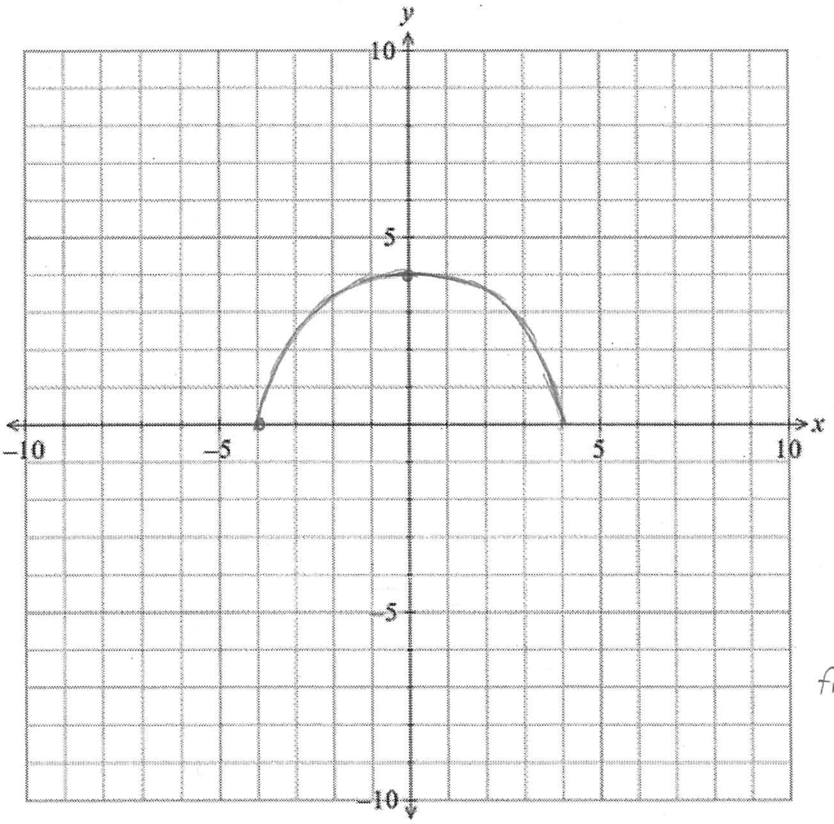
$-ax^2 + c - 9 \geq 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\pm\sqrt{-4(-a)(c-9)}}{-2a}$

$a = -a$
$b = 0$
$c = c - 9$

$x = \frac{\pm\sqrt{4a(c-9)}}{2a}$

(e) Sketch $f(g(x))$ if $a = 1$ and $c = 25$. (2 marks)



$f(g(x))$ when $a = 1$
$c = 25$

$y = \sqrt{-x^2 + 16}$

$y^2 = -x^2 + 16$

$x^2 + y^2 = 16$

Circle centre $(0,0)$
$r = 4$

from c) and d)

Domain $-4 \leq x \leq 4$

Range $0 \leq y \leq 4$

END OF QUESTIONS

✓ $\frac{1}{2}$ circle

✓ over correct domain & range.

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Test 1, 2023
Section Two: Calculator Allowed
Complex Numbers and Functions

STUDENT'S NAME: ______________________________

DATE: Thursday 16th March **TIME:** 10 minutes **MARKS:** 10
ASSESSMENT %: 10

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser
Special Items: 1 A4 page notes, Classpad, Scientific Calculator

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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Question 7 **(10 marks)**

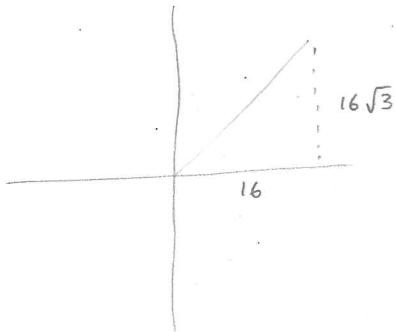
Consider the equation $z^5 = 16 + 16\sqrt{3}i$, where $z \in \mathbb{C}$.

(a) Give exact solutions to the equation in the form $r cis\theta$ where $0 \le \theta \le 2\pi$. (4 marks)

$z^5 = 16 + 16\sqrt{3}\, i$

$z^5 = 32 \text{ cis} \left(\frac{\pi}{3} + 2\pi k \right)$

$z_k = 32^{1/5} \text{ cis} \left(\frac{\pi}{15} + \frac{6\pi k}{15} \right)$



$|z| = \sqrt{16^2 + (16\sqrt{3})^2}$

$= \sqrt{16^2(1+3)}$

$= 16 \times 2 = 32$

$\tan\theta = \frac{16\sqrt{3}}{16} = \sqrt{3}$

$\theta = \frac{\pi}{3}$

Spaced $\frac{2\pi}{5} = \frac{6\pi}{15}$

$k = 0 \quad z_0 = 2 \text{ cis } \pi/15$

$k = 1 \quad z_1 = 2 \text{ cis } 7\pi/15$

$k = 2 \quad z_2 = 2 \text{ cis } 13\pi/15$

$k = 3 \quad z_3 = 2 \text{ cis } 19\pi/15$

$k = 4 \quad z_4 = 2 \text{ cis } 25\pi/15$

$(2 \text{ cis } 5\pi/3)$

✓ converts to $r.cis\,\theta$

✓ de Moivre's

✓ 5 solutions

✓ over correct $0 \le \theta \le 2\pi$

If w is the solution with the smallest argument and u is the solution with the largest argument:

(b) State w and u. (1 mark)

$w = 2 \text{ cis } \pi/15$

$u = 2 \text{ cis } 25\pi/15$

✓ states both w & u correctly

(c) Determine $Re(wu)$. (2 marks)

$wu = 4 \text{ cis } 26\pi/15$

$Re(wu) = 4 \cos 26\pi/15 = 2.68$ (2dp)

✓ wu

✓ Re part

(d) Determine $|3w^3|$. (1 mark)

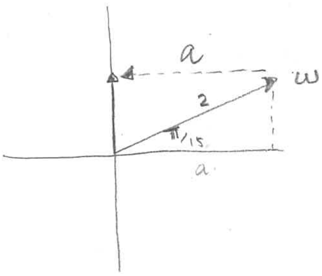
$w^3 = 8 \text{ cis } \frac{3\pi}{15}$

$3w^3 = 24 \text{ cis } \frac{3\pi}{15}$

✓ correct $|3w^3|$

$|3w^3| = 24$

(e) Determine a if $a \in \mathbb{R}$ and $Arg(w + a) = \frac{\pi}{2}$. (2 marks)



$a = -Re(w)$

$= -(2 \cos \pi/15)$

$a = -1.96$

✓ $-Re(w)$

✓ states $a =$

END OF QUESTIONS